luh-ploss

7.1 - Definition of the Laplace Transform

A tool for solving differential equations

Definition: A **transform** transforms one function to another function.

Examples of transforms
The derivative and the integral:

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ and $\frac{d}{dx}(x^3) = 3x^2$ are transforms.

$(x+y)^2$

$\int \frac{1}{x}\,dx = \ln|x| + C$ is a transform.

$\sin(3x+2y)$

Transforms possess the linearity properties, e.g.,
$\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)]$

Definition 7.1.1: Let $f(t)$ be a function defined for $t \geq 0$. Then the integral $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ is said to be the **Laplace transform** of f, provided that the integral converges.

part of the def. It's always there

this is what's being transformed

The Laplace transform possesses the linearity properties.

$f(t)=1$

$\cancel{\lim_{b\to\infty}\int_0^b e^{-st}\,dt}$

Example 1: Evaluate $\mathscr{L}\{1\}$.

$\mathscr{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt$

$u=-st$

$du=-s\,dt$

$= -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$

0

$\boxed{\mathscr{L}\{1\} = \frac{1}{s}}, \quad s>0$

Example 2: Evaluate $\mathscr{L}\{t\}$.

$$\mathcal{L}\{t\} = \int_0^\infty t\, e^{-st}\, dt \qquad u = t \quad dv = e^{-st}dt$$

$$du = dt \quad v = -\frac{1}{s}e^{-st}$$

$$= \cancel{-\frac{t}{se^{st}}\Big|_0^\infty} + \int_0^\infty \frac{1}{s}e^{-st}\,dt$$

$$= -\frac{1}{s^2}e^{-st}\Big|_0^\infty = \frac{1}{s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \quad , s > 0$$

Example: Evaluate $\mathscr{L}\{4t - 7\}$

$$\mathcal{L}\{4t-7\} = 4\mathcal{L}\{t\} - 7\mathcal{L}\{1\}$$

$$= \boxed{\frac{4}{s^2} - \frac{7}{s}}$$

Example: Evaluate $\mathscr{L}\{\sin kt\}$

$$\mathcal{L}\{\sin kt\} = \int_0^\infty e^{-st}\sin kt\, dt = I$$

$$u = \sin kt \qquad dv = e^{-st}dt$$

$$du = k\cos kt\, dt \qquad v = -\frac{1}{s}e^{-st}$$

$$I = \cancel{-\frac{1}{se^{st}}\sin kt\Big|_0^\infty} + \int_0^\infty \frac{k}{s}e^{-st}\cos kt\, dt$$

$$u = \frac{k}{s}\cos kt \qquad dv = e^{-st}dt$$

$$du = -\frac{k^2}{s}\sin kt\, dt \qquad v = -\frac{1}{s}e^{-st}$$

$$I = -\frac{k}{s^2e^{st}}\cos kt\Big|_0^\infty - \frac{k^2}{s^2}I$$

$$\left(\frac{k^2}{s^2}+1\right)I = \frac{k}{s^2} \Rightarrow \frac{k^2+s^2}{s^2} I = \frac{k}{s^2}$$

$$I = \mathcal{L}\{\sin kt\} = \frac{k}{k^2+s^2}$$

Notation: $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$

Linearity:

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

Theorem 7.1.1: Transforms of Some Basic Functions

$$\text{(a) } \mathscr{L}\{1\} = \frac{1}{s}$$

(b) $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $n = 1,\ 2,\ 3,\ \ldots$

(c) $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

(e) $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$

Definition: A function f is said to be of **exponential order** if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

Theorem: Sufficient Conditions for Existence

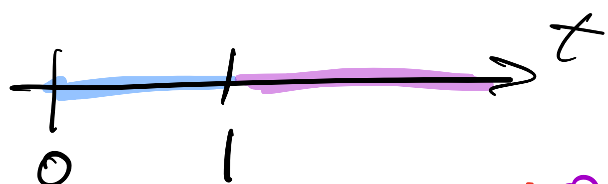
If f is piecewise continuous on $[0,\ \infty)$ and of exponential order, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

Theorem: Behavior of $F(s)$ as $s \to \infty$

If f is piecewise continuous on $[0, \infty)$ and of exponential order and $F(s) = \mathscr{L}\{f(t)\}$, then $\lim_{s \to \infty} F(s) = 0$.

Example: Use Definition 7.1.1 to find $\mathscr{L}\{f(t)\}$.

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$



$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}(2t+1)\,dt + \cancel{\int_1^\infty e^{-st}(0)\,dt}$$

$$u = 2t+1 \qquad dv = e^{-st}\,dt$$

$$du = 2dt \qquad v = -\frac{1}{s}e^{-st}$$

$$F(s) = \left(-\frac{2t+1}{s e^{st}} - \frac{2}{s^2 e^{st}} \right|_0^1$$

$$F(s) = -\frac{3}{s e^s} - \frac{2}{s^2 e^s} + \frac{1}{s} + \frac{2}{s^2}$$

$$= \boxed{\frac{1}{s^2}\left(2 - 3e^{-s}\right) + \frac{1}{s}\left(1 - 2e^{-s}\right)}$$

Example: Use Definition 7.1.1 to find $\mathscr{L}\{f(t)\}$.

$f(t) = e^{-2t-5}$

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} e^{-2t-5}\, dt = e^{-5}\int_0^\infty e^{-(s+2)t}\, dt$$

$$= e^{-5}\left(-\frac{1}{s+2} e^{-(s+2)t}\right|_0^{\infty}$$

$$= \boxed{\frac{e^{-5}}{s+2}}$$

$$\int \cos^2 x\,dx = \int \tfrac{1}{2}(1+\cos 2x)\,dx$$

(e) $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$

Example: Use Theorem 7.1.1 to find $\mathscr{L}\{f(t)\}$.

$f(t) = t^2 - e^{-9t} + 5$

$$\boxed{F(s) = \frac{2}{s^3} - \frac{1}{s+9} + \frac{5}{s}}$$

$f(t) = \cos^2 t$

$$f(t) = \tfrac{1}{2} + \tfrac{1}{2}\cos 2t$$

$$\boxed{F(s) = \frac{1}{2s} + \frac{s}{2(s^2+4)}}$$

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, $n = 1, 2, 3, \ldots$

(c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$